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NOTE

On the Question about the Sufficiency of the von Neumann Criterion for Stability of Difference Schemes

1. INTRODUCTION

The Fourier method [1] is the most popular practical method for stability investigation of difference schemes. The von Neumann condition, which is necessary for the stability of a difference scheme, has the form

$$\max_i |\lambda_i(\mathbf{k})| \leq 1 + O(\tau) \qquad \text{at all } \mathbf{k}, \tag{1}$$

where λ_i are the eigenvalues of the amplification matrix G of a difference scheme, τ is the time step, and $\mathbf{k}$ is a real wave vector. It is well known that the von Neumann condition is not only necessary but also sufficient if the amplification matrix G is a normal matrix (or, what is the same, the step operator S is a normal operator):

$$G^* G = G G^* \qquad (S^* S = S S^*). \tag{2}$$

The relationship

$$\|S\| = \max_{\mathbf{k}} \|G(\mathbf{k})\| = \max_{\mathbf{k}} \max_i |\lambda_i(\mathbf{k})| \tag{3}$$

is valid for the normal matrix G, which ensures the sufficiency of the von Neumann condition for the stability of the difference scheme.

We present in the following new, more general conditions at which the von Neumann criterion is not only necessary but also sufficient. We have proved, in particular, that the von Neumann condition is sufficient if the matrix G^N is a normal matrix, where N is any finite exponent.

2. THE CASE OF THE TWO-CYCLE MACCORMACK SCHEME

The stability of the two-cycle MacCormack scheme for the two-dimensional advection equation

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x_1} + B\frac{\partial u}{\partial x_2} = 0$$

was considered in [2], where A and B are scalar constants. The stability region of the difference scheme under consideration was determined in the plane of nondimensional parameters κ_1 and κ_2,

$$\kappa_1 = A\tau/h_1, \qquad \kappa_2 = B\tau/h_2,$$

where h_1 and h_2 are the steps of a uniform grid along the x_1- and x_2-axes, respectively. An exact boundary of the region inside which the von Neumann criterion is satisfied was obtained on the basis of the catastrophe theory methods. The step operator S for the given scheme has the form

$$S = \begin{pmatrix} 0 & 1 \\ L_2 L_1 & 0 \end{pmatrix}, \tag{4}$$

where L_1 and L_2 are certain difference operators. The Fourier transform of the operator S determines the amplification matrix

$$G = \begin{pmatrix} 0 & 1 \\ a+ib & 0 \end{pmatrix}, \tag{5}$$

where $a + ib$ is the Fourier transform of the operator $L_2 L_1$. It is easy to see that the step operator S as well as the amplification matrix G does not satisfy condition (2). Therefore, it was noted in [2] that the region found in the (κ_1, κ_2) plane is a region in which the necessary conditions for the stability of a difference scheme are satisfied.

We now prove that the von Neumann criterion for the two-cycle MacCormack difference scheme considered in [2] is not only the necessary but also sufficient stability condition. Thus, the above difference scheme is stable at all values of the parameters κ_1, κ_2, which lie inside the region obtained in [2].

It is easy to see that the squared operator S,

$$S^2 = L_2 L_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

as well as the corresponding squared amplification matrix G,

$$G^2 = (a+ib) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is a normal operator. Therefore, the equality

$$\|S^2\| = \max_{\mathbf{k}} \max_i |\zeta_i(\mathbf{k})|$$

is valid, where ζ_i are the eigenvalues of the matrix G^2. It is well known from the theory of linear operators (see, e.g., [3]) that at all $\mathbf{k}$

$$\zeta_i = \lambda_i^2,$$

where the λ_i are the eigenvalues of the matrix G. Therefore, in the region where the von Neumann condition is satisfied (with $O(\tau) = 0$) the estimate of the form

$$\|S^2\| = \max_{\mathbf{k}} \left|\lambda_i^2(\mathbf{k})\right| \le 1 \tag{6}$$

holds.

Let us consider the original stability definition of a difference scheme [4].

A difference scheme is called stable, if there exists the number $M(\overline{t}) > 0$ such that

$$\|C_{n,k}\| \leq M(\overline{t}) \tag{7}$$

for all $0 \leq k \leq n-1$, $t_n \leq \overline{t}$. Here $C_{n,k}$ is the transition operator for a passage from the kth time level to the nth time level and $t_n = \tau_1 + \tau_2 + \cdots + \tau_n$, where τ_k is the time step of a finite difference scheme.

Since the difference scheme has constant coefficients in our case, $C_{n,k} = S^{n-k}$. With regard for inequality (6) we can obtain the following estimate for the norm of S^{n-k} at any n and k $(0 \leq k < n)$,

$$\|S^{n-k}\| \leq \|S^{2p+\delta}\| \leq \|S^2\|^p \|S\|^\delta \leq \|S\|^\delta, \tag{8}$$

where $p = [\frac{n-k}{2}]$ (the integral part of $\frac{n-k}{2}$), and

$$\delta = (n-k) - 2\left[\frac{n-k}{2}\right] = \begin{cases} 1 & \text{for odd } (n-k) \\ 0 & \text{for even } (n-k). \end{cases}$$

Then we obtain from (8) that

$$\|C_{n,k}\| = \|S^{n-k}\| \leq M, \tag{9}$$

where $M = \max\{1, \|S\|\}$.

It is easy to see that

$$\|S\| = \max_{\mathbf{k}} \|G(\mathbf{k})\| \leq \max_{\xi} \max_{i,j} |g_{i,j}(\xi, \kappa_1, \kappa_2)| \leq M_1(\kappa_1, \kappa_2),$$

where $g_{i,j}$ are the elements of the matrix G (5), and $\xi = (k_1 h_1, k_2 h_2)$ is the vector of spectral parameters. At any values of κ_1 and κ_2 from the region in which the von Neumann criterion is satisfied the quantity M_1 is bounded. Consequently, the inequality (9) proves the stability of the two-cycle MacCormack scheme considered in [2].

3. GENERAL CASE

The following theorem is valid in a general case.

THEOREM. *Let the following two conditions be satisfied.*

(1) *The estimate*

$$\|C_{n,k}\| \leq \|S^{n-k}\|$$

holds for the norm of the transition operator $C_{n,k}$ *of a difference scheme, where* S *is some majorant step operator with constant coefficients.*

(2) *For the Fourier transform* $G(\mathbf{k})$ *of operator* S *there exists a certain integer* $N > 0$ *such that* G^N *is a normal matrix.*

(3) *For some positive* $\overline{\tau}$ *the operator* S *is uniformly bounded,*

$$\|S\| \leq M_1 \qquad \textit{at } 0 < \tau_k < \overline{\tau}\ (0 \leq k \leq n-1).$$

Then the von Neumann condition for the operator S is the necessary and sufficient stability condition of a difference scheme.

Proof. Let us denote by $\zeta_i(\mathbf{k})$ the eigenvalues of the matrix G^N. It is known that

$$\zeta_i(\mathbf{k}) = \lambda_i^N(\mathbf{k}), \tag{10}$$

where λ_i are the eigenvalues of the matrix G. Let the von Neumann condition be satisfied,

$$\max_i |\lambda_i(\mathbf{k})| \leq 1 \qquad \text{at all } \mathbf{k} \tag{11}$$

(the proof below can be easily generalized to the case when $O(\tau) \neq 0$ in (1)). Then the following estimate holds

$$\|S^N\| = \max_{\mathbf{k}} \|G^N(\mathbf{k})\| = \max_{\mathbf{k}} \max_i |\zeta_i(\mathbf{k})| = \max_{\mathbf{k}} \max_i |\lambda_i(\mathbf{k})|^N \leq 1. \tag{12}$$

Let us estimate the norm of the transition operator $C_{n,k}$ with regard for condition (1) of the Theorem and inequality (12),

$$\|C_{n,k}\| \leq \|S^{n-k}\| = \|S^{N_{p+\delta}}\| \leq \|S^N\|^p \|S\|^\delta \leq \|S\|^\delta, \tag{13}$$

Where $p = [\frac{n-k}{N}]$ and $\delta = (n-k) - N[\frac{n-k}{N}]$. It is easy to see that

$$\|S\|^\delta \leq M_2, \tag{14}$$

where $M_2 = \max\{1, \|S\|^{N-1}\}$. It follows from condition (3) of the Theorem that M_2 is a bounded quantity, which does not depend on the time steps τ_k:

$$M_2 \leq \max\left\{1, M_1^{N-1}\right\}.$$

Substituting (14) into (13) we obtain that

$$\|C_{n,k}\| \leq M, \tag{15}$$

where M does not depend on n and k.

Inequality (15) coincides with the stability definition of difference scheme (7). The theorem is proved.

Remark. If the difference scheme coefficients are constant quantities, then the majorant operator S coincides with the step operator.

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